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Supergravity backgrounds corresponding to D7 branes wrapped on Kähler manifolds

Martin Kruczenski

Department of Physics, Brandeis University
Waltham, MA 02454, U.S.A.
E-mail: martink@brandeis.edu

Abstract: We consider supergravity solutions corresponding to D7 branes wrapped on Kähler manifolds with a $\mathrm{U}(1)_R$ twist such that some supersymmetry is preserved. We find a class of 1/4-BPS backgrounds where a $D7$-brane is wrapped on a T^2 torus with a metric of non-constant curvature. Similarly to the flat $D7$-brane case, the solution has a singularity at finite radius. We also discuss the case where the $D7$-brane is wrapped on a 4-dimensional non-compact manifold. The field theories on the D7 brane have $\mathcal{N} = 1$ supersymmetry in 6 and 4 dimensions respectively.

Keywords: F-Theory, D-branes, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

The discovery of D-branes [1] led to an enormous progress in string theory. At first, the interest was that they were the long sought string theory objects carrying RR-charge. However, it was soon understood that a perhaps more important role they had was to allow the construction of many different gauge theories as particular limits of string theory. Previously, that was only possible by using the heterotic string. Now, the world-volume theories on the D-branes were gauge theories with various matter contents (depending on the type of D-brane or D-brane intersection etc.) and many important properties of the gauge theories were understood as simple geometrical or physical properties of D-branes.[1]

For the purpose of this paper it is of particular interest the construction that makes use of D7 branes wrapped on Kähler manifolds [3]. The idea is to wrap a D7 brane on a $2p$ dimensional complex manifold of $\mathrm{U}(p) = \mathrm{SU}(p) \times \mathrm{U}(1)$ holonomy (Kähler manifold) times $(8-2p)$-dimensional flat space. If the U(1) part of the holonomy does not vanish, then there are no covariantly constant spinors and supersymmetry is completely broken. However, the theory has an $\mathrm{U}(1)_R$ R-symmetry under which the fermions transform non-trivially. It is possible to gauge the $\mathrm{U}(1)_R$ and introduce a fixed connection in such a way that it cancels the U(1) part of the holonomy. This is known as twisting the theory. The resulting theory preserves one supercharge in $8-2p$ dimensions. Note that this is

[1] See [2] for a review.

different from the case where the holonomy is SU(p) (Calabi-Yau manifold) since there two supercharges are preserved. They have opposite charges under the U(1) but the same under the U(1)$_R$ so, if the U(1) is non-trivial only one of them can be preserved by twisting.

The supergravity background corresponding to such construction should have a non-trivial metric, axion and dilaton. The axion and dilaton can be lumped together in the complex field

$$\tau = \chi + ie^{-\phi} \,. \tag{1.1}$$

Type IIB theory is invariant under an SL$(2, \mathbb{Z})$ duality which acts on τ as

$$\tau \to \frac{a\tau + b}{c\tau + d} \,, \qquad a, b, c, d \in \mathbb{Z} \,, \qquad ab - cd = 1 \,. \tag{1.2}$$

It follows that τ does not need to be a globally well defined function, but that in different coordinate patches one can have different functions as long as the relation between them is as in (1.2).

As we review later in the paper, some amount of supersymmetry is preserved if we consider the metric to be of the type

$$ds^2 = ds^2_{[1,9-2p]} + \partial_{a\bar{b}} K\left(z^a, \bar{z}^{\bar{b}}\right) \, dz^a d\bar{z}^{\bar{b}} \,, \tag{1.3}$$

where $ds^2_{[1,9-2p]}$ is a flat metric of signature $[1, 9-2p]$. The z^a, $(a = 1 \ldots p)$ are complex coordinates and the Kähler potential $K(z^a, \bar{z}^{\bar{b}})$ has to satisfy

$$\det\left[\partial_{a\bar{b}} K\left(z^a, \bar{z}^{\bar{b}}\right)\right] = \Omega(z^a)\bar{\Omega}(z^{\bar{b}}) \frac{\left(\tau\left(z^a\right) - \bar{\tau}\left(\bar{z}^{\bar{b}}\right)\right)}{2i} \,, \tag{1.4}$$

with $\Omega(z^a)$ and $\tau(z^a)$ arbitrary holomorphic functions except for the fact that $\mathrm{Im}(\tau) = e^{-\phi} > 0$.

One further point is that τ can be thought of as the modular parameter of a flat torus T^2. By incorporating this torus explicitly, one can write down a 12-dimensional Ricci flat metric such that the Kähler manifold and the torus give rise to a $p+1$ complex dimensional Calabi-Yau manifold. This constructions is known as F-theory [4] and from this point of view, the D7-branes are Calabi-Yau compactifications of F-theory.

In the rest of the paper we consider the cases $p = 1, 2$ and find non trivial solutions to (1.4). The metric (1.3) thus obtained has curvature singularities in the same way as in the flat $D7$-brane ($p = 0$) case.

2. D7-brane solutions

D7-brane solutions were pioneered in [5] although in a different context. What was found there are string theory compactification on elliptically fibered Calabi-Yau manifolds. The Calabi-Yau can be described as a certain (Kähler) base space $\mathcal{B}$ over which a torus of modular parameter τ is fibered. By that, one means that τ is a given (holomorphic) function on the base space. The construction made explicit use of the fact that such function τ only needs to be defined up to modular transformations to construct supergravity solutions such that τ had non-trivial monodromies around the 'core' of the solution. This is in fact a generalization of the more familiar construction of vortices in superconductors or cosmic strings.

It was realized in [6] that the same construction could be made in type IIB theory since, due to the $\mathrm{SL}(2,\mathbb{Z})$ invariance of type IIB, the field $\tau = \chi + i\, e^{-\phi}$ is also defined up to a modular transformation. The solution represents IIB theory compactified on $\mathcal{B}$.

Afterwards, another D7-brane solution was found in [7] and described as a circularly symmetric D7-brane. Later, in [8] it was realized that they were both part of the same construction. The idea was that one can find a solution of type IIB supergravity with metric

$$ds^2 = dx^2_{[1,7]} + \Omega(z)\bar{\Omega}(\bar{z})\tau_2 dz d\bar{z}\,, \tag{2.1}$$

and with Ω and $\tau(z) = \tau_1 + i\tau_2$ arbitrary[2] holomorphic functions of z. This is a solution which locally preserves 16 real supercharges of IIB. Different choices of the functions functions lead to different solutions. Note that all these solutions have curvature singularities for certain values of z.

3. Wrapped D7-branes

In this paper we consider solutions describing wrapped D7-branes. These solutions will have only excited the axidilaton $\tau = \chi + i\, e^{-\phi}$ and the metric. It is known that when wrapping D-branes on curved manifolds, charges corresponding to lower dimensional branes may arise. Here, we will consider that if that is the case, then corresponding lower dimensional branes have been included to cancel the induced charge.

Under these circumstances, the equations of motion (in Einstein frame) that we want to solve are [10]:

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\left(\partial_\mu\phi\partial_\nu\phi + e^{2\phi}\partial_\mu\chi\partial_\nu\chi\right), \\ \triangle\phi &= e^{2\phi}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi\,, \\ \triangle\chi &= -2g^{\mu\nu}\partial_\mu\phi\partial_\nu\chi\,, \end{aligned} \tag{3.1}$$

where $\triangle$ denotes the covariant laplacian:

$$\triangle\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right). \tag{3.2}$$

In terms of τ they can be written as

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{4\tau_2^2}\left(\partial_\mu\tau\partial_\nu\bar{\tau} + \partial_\nu\tau\partial_\mu\bar{\tau}\right), \\ \triangle\tau &= \frac{2}{\tau - \bar{\tau}}g^{\mu\nu}\partial_\mu\tau\partial_\nu\tau\,. \end{aligned} \tag{3.3}$$

[2] Actually there are some global restrictions if supersymmetry is to be preserved. See [9, section 3.2] for a discussion. I am grateful to Ulf Gran for pointing out this to me.

Notice, for later reference, that the equations are invariant under the rescaling

$$g_{\mu\nu} \to \lambda g_{\mu\nu}\,, \qquad \tau \to \sigma\tau\,, \tag{3.4}$$

for arbitrary constant λ, σ.

The solution we are seeking should be similar to (2.1) but with $dx^2_{[1,7]}$ replaced by $dx^2_{[1,7-2p]} + ds^2_{[2p]}$ with $ds^2_{[2p]}$ the metric of the Kähler manifold where the D7-brane is wrapped. This, in fact, cannot be the whole story since the metric $ds^2_{[2p]}$ can have parameters which depend on the transverse coordinates and we also have to incorporate the twist that is used to preserve supersymmetry. Instead of searching for an ansatz with this properties, it is simpler to start by analyzing the conditions under which the right amount of supersymmetry is preserved. We use an ansatz for the metric

$$ds^2 = dx^2_{[1,7-2p]} + g_{a\bar{b}} dz^a d\bar{z}^{\bar{b}}\,, \tag{3.5}$$

where $g_{a\bar{b}}$ is a Kähler metric with Kähler potential K: $g_{a\bar{b}} = \partial_{a\bar{b}} K$. This metric and the supersymmetry variations that we compute below are in Einstein frame. Introducing a vielbein e^A_a, $\bar{e}^{\bar{A}}_{\bar{a}}$, the dilatino variation is given by [10][3]

$$\delta\lambda = -\frac{1}{2\tau_2}\left(\partial_a \tau e^a_A \Gamma^A + \bar{\partial}_{\bar{a}}\tau e^{\bar{a}}_{\bar{A}}\Gamma^{\bar{A}}\right)\varepsilon^*\,. \tag{3.6}$$

If we now consider a spinor ε annihilated by the $\Gamma^{\bar{A}}$:

$$\Gamma^{\bar{A}}\varepsilon = 0\,, \qquad \Gamma^A\varepsilon^* = 0\,, \tag{3.7}$$

then we have that $\delta\lambda = 0$ is satisfied if τ is holomorphic, namely $\bar{\partial}_{\bar{a}}\tau = 0$. On the other hand, the gravitino variation is [10, 7]

$$\begin{aligned} \delta\Psi_a &= \partial_a\varepsilon + i\omega_{aA\bar{B}}\Sigma^{A\bar{B}} + i\omega_{a\bar{A}B}\Sigma^{\bar{A}B} + \frac{i}{4}\frac{\partial_a\tau_1}{\tau_2}\varepsilon\,, \\ &= \partial_a\varepsilon + \frac{1}{4}\omega_{aA\bar{B}}[\Gamma^A,\Gamma^{\bar{B}}]\varepsilon + \frac{1}{4}\omega_{a\bar{A}B}[\Gamma^{\bar{A}},\Gamma^B]\varepsilon + \frac{i}{4}\frac{\partial_a\tau_1}{\tau_2}\varepsilon\,, \end{aligned} \tag{3.8}$$

where we used that $\Sigma^{A\bar{B}} = -\frac{i}{4}[\Gamma^A,\Gamma^{\bar{B}}]$. Using the (anti)commutation properties of the Dirac matrices $\{\Gamma^A,\Gamma^{\bar{B}}\} = \delta^{A\bar{B}}$ and (3.7) we can recast this as

$$\delta\Psi_a = \partial_a\varepsilon - \frac{1}{2}\sum_A \omega_{aA\bar{A}}\varepsilon + \frac{i}{4}\frac{\partial_a\tau_1}{\tau_2}\varepsilon\,. \tag{3.9}$$

Now, from appendix B, we know that, for a Kähler manifold,

$$\sum_A \omega_{aA\bar{A}} = -\partial_a \ln\det \bar{e}^{\bar{B}}_{\bar{b}}\,, \tag{3.10}$$

and so the gravitino variation vanishes if

$$\partial_a \ln\det \bar{e}^{\bar{B}}_{\bar{b}} = -\frac{i}{2}\frac{\partial_a\tau_1}{\tau_2} = \frac{1}{2}\partial_a \ln\tau_2\,, \tag{3.11}$$

[3] We actually follow a simpler version which can be found in [7].

where we used $\partial_a \bar{\tau} = 0$ and the ansatz $\partial_a \varepsilon = 0$. Eq. (3.11) is solved if

$$\det \bar{e}_{\bar{a}}^{\bar{A}} = \bar{\Omega}(\bar{z}^{\bar{a}})\sqrt{\tau_2}\,, \tag{3.12}$$

with an arbitrary holomorphic function $\Omega(z^a)$. From here we obtain

$$\det g_{a\bar{b}} = \det e_a^A \det \bar{e}_{\bar{b}}^{\bar{B}} = \Omega(z^c)\bar{\Omega}(\bar{z}^{\bar{c}})\tau_2\,. \tag{3.13}$$

We can always eliminate locally the function Ω by an appropriate change of coordinates. Thus, we find that we need a Kähler manifold which admits a Kähler potential such that

$$\det g_{a\bar{b}} = \mathrm{Im}\,[\tau(z^c)]\,, \tag{3.14}$$

in each coordinate patch and for a given holomerphic function $\tau(z^c)$. Notice that this has similarities with the Calabi-Yau case where the condition is that $\det g = 1$ locally. In both cases, this implies one equation for one indeterminate K which in principle can be solved up to global obstructions. Instead of further analyzing the mathematical properties of the metrics satisfying equation (3.14), in this paper we limit ourselves to look for examples.

Before doing so, however, we should check that the equations of motion are also satisfied. It is easy to see that the axidilaton equation in (3.3) is satisfied if τ is holomorphic. In that case the equation for the metric reduces to

$$R_{a\bar{b}} = \frac{1}{4}\frac{\partial_a \tau \bar{\partial}_{\bar{b}} \bar{\tau}}{\tau_2^2}\,. \tag{3.15}$$

If we now use that

$$\begin{aligned} R_{a\bar{b}} &= -\partial_{a\bar{b}} \ln \det g_{c\bar{d}}\,, \\ \partial_{a\bar{b}} \ln \tau_2 &= -\frac{1}{4}\frac{\partial_a \tau \partial_{\bar{b}} \bar{\tau}}{\tau_2^2}\,, \end{aligned} \tag{3.16}$$

eq. (3.15) becomes

$$-\partial_{a\bar{b}} \ln \det g_{c\bar{d}} = -\partial_{a\bar{b}} \ln \tau_2\,, \tag{3.17}$$

which is satisfied in view of (3.13).

It was observed in [5] that, starting from a $2p$-dimensional metric that satisfies (3.13) one can construct a $2(p+1)$-dimensional Ricci flat metric. Introducing an extra complex coordinate ζ and defining a new Kähler potential as

$$\mathcal{K}(z^a, \zeta, \bar{z}^{\bar{b}}, \bar{\zeta}) = K(z^a, \bar{z}^{\bar{b}}) - \frac{(\zeta - \bar{\zeta})^2}{\tau_2}\,, \tag{3.18}$$

one gets a metric

$$ds^2 = \partial_{a\bar{b}} K dz^a d\bar{z}^{\bar{b}} + \frac{1}{\tau_2}\left| d\zeta - \frac{\zeta - \bar{\zeta}}{2i}\frac{\partial_a \tau dz^a}{\tau_2}\right|^2\,. \tag{3.19}$$

From here it follows that

$$\det g_{i\bar{j}} = \frac{1}{\tau_2} \det \partial_{a\bar{b}} K\,, \tag{3.20}$$

where $i = a, \zeta$ and $\bar{\jmath} = \bar{b}, \bar{\zeta}$. Using (3.13) this gives

$$\det g_{i\bar{\jmath}} = |\Omega(z^a)|^2 \,. \tag{3.21}$$

Since $R_{i\bar{\jmath}} = -\partial_{i\bar{\jmath}} \ln \det g_{k\bar{l}}$ we get $R_{i\bar{\jmath}} = 0$, namely the metric is Ricci flat. This means that the manifold is a Calabi-Yau manifold. Although the metric so constructed turns out to be singular at certain points, as explained in [5] it can be deformed to a non-singular one. In the context of type IIB theory in which we are working this construction is called a compactification of F-theory [4] on the (elliptically fibered) Calabi-Yau of Kähler potential $\mathcal{K}$. The base of the fibration $\mathcal{B}$ is the manifold we are studying.

4. Complex dimension 1 (8 real supercharges)

In this section we concentrate on the case $p = 1$, namely, when wrapping the D7 brane on a 2-dimensional manifold. The resulting theory on the D7 brane is a 6-dimensional field theory with 8 supercharges. By further compactification on a flat T^2 torus we get an $\mathcal{N} = 2$ four dimensional theory.

In view of the discussion of the previous section, we are looking for a function $K(z_1, z_2, \bar{z}_1, \bar{z}_2)$ such that

$$\det \partial_{a\bar{b}} K = \Omega(z^c)\bar{\Omega}(\bar{z}^{\bar{c}})\tau_2 \,. \tag{4.1}$$

Since τ is a holomorphic non-constant function we can locally use it as a coordinate and identify $\tau = z_2$. The equation then becomes

$$K_{1\bar{1}}K_{2\bar{2}} - K_{1\bar{2}}K_{2\bar{1}} = |\Omega|^2 \frac{(z_2 - \bar{z}_2)}{2i} \,. \tag{4.2}$$

Although simple looking, this is a non-linear equation in partial derivatives which is very difficult to solve in full generality. The idea here is to find particular solutions using simplifying assumptions. The first such assumption is that K depends only on the modulus of z_1. This implies that there is an U(1) isometry $z_1 \to e^{i\theta} z_1$. Actually, for the purpose of the calculation it is easier to do a coordinate transformation $w_1 = \ln z_1$ and then K will depend only on $w_1 + \bar{w}_1 = \ln |z_1|^2$. Equivalently, we can consider that K depends only on the real part of z_1. We shall see that this assumption makes eq. (4.2) simpler to solve. A second and less necessary assumption is that K depends only on τ_2, the imaginary part of z_2. Introducing real coordinates through

$$z_1 = y + i\phi \,, \qquad z_2 = \tau = \theta + ix \,, \tag{4.3}$$

the assumption is that $K(z^a, \bar{z}^{\bar{a}}) = K(x, y)$. Thus, eq. (4.2) reduces to:

$$K_{yy}K_{xx} - K_{xy}^2 = x \,, \tag{4.4}$$

where we did the final assumption that $\Omega = \text{cst.} = 1/4$. Notice that in principle we could have had $|\Omega| = e^{\alpha x + \beta y}$. The assumption is that $\alpha = \beta = 0$. The equation is still a non-linear equation in partial derivatives but now depending only on two variables. It can be solved by means of a Legendre transformation. Notice that if in the right hand side we would have had 1 instead of x, then we would have been looking for a hyperKähler manifold and the Legendre transform is a well-known method [11].

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The Legendre transform method starts by defining a new potential $\tilde{K}$ through

$$\tilde{K}(\eta, x) = K(y, x) - \eta y\,, \qquad \eta = K_y\,, \tag{4.5}$$

which implies

$$\tilde{K}_\eta = -y\,, \qquad \tilde{K}_x = K_x\,, \qquad K_{xx} = \tilde{K}_{xx} - \frac{\tilde{K}^2_{x\eta}}{\tilde{K}_{\eta\eta}}\,, \qquad K_{yy} = -\frac{1}{\tilde{K}_{\eta\eta}}\,, \qquad K_{xy} = -\frac{\tilde{K}_{x\eta}}{\tilde{K}_{\eta\eta}}\,. \tag{4.6}$$

Using these properties eq. (4.4) becomes

$$\tilde{K}_{xx} + x\tilde{K}_{\eta\eta} = 0\,, \tag{4.7}$$

which is now a linear differential equation. A particularly simple solution depending on one parameter ($a_0 > 0$) is

$$\tilde{K} = a_0(x^3 - 3\eta^2)\,, \tag{4.8}$$

which as we check below corresponds to a flat D7-brane. It is not much more difficult to find the most general solution (see appendix A for a derivation):

$$\tilde{K}(\eta, x) = a_0(x^3 - 3\eta^2) - \frac{3^{1/3}}{2\pi^2}\Gamma\left(\frac{2}{3}\right)^3 \int_{-\infty}^{+\infty} \frac{x}{\left[(\eta - \tilde{\eta})^2 + \frac{4}{9}x^3\right]^{5/6}} h(\tilde{\eta})d\tilde{\eta}\,, \tag{4.9}$$

which depends on an arbitrary function $h(\eta)$ that, as we shall see, determines the metric in the manifold over which the D7 is wrapped. The derivation of this expression is given in the appendix A. However it is easy to check that it solves eq. (4.7) since

$$(\partial_{xx} + x\partial_{\eta\eta}) \frac{x}{\left[\eta^2 + \frac{4}{9}x^3\right]^{5/6}} = 0\,. \tag{4.10}$$

The coefficient in front of the integral in (4.9) is chosen such that

$$\tilde{K}(x = 0, \eta) = -3a_0\eta^2 - h(\eta)\,. \tag{4.11}$$

From here we can undo the Legendre transformation and get the Kähler potential if we wished. However it is simpler to use η as a coordinate and get the metric directly. To see that, we start by writing the metric in the original coordinates

$$\begin{aligned} ds^2 &= K_{1\bar{1}}dz_1 d\bar{z}_1 + K_{1\bar{2}}dz_1 d\bar{z}_2 + K_{\bar{1}2}d\bar{z}_1 dz_2 + K_{2\bar{2}}dz_2 d\bar{z}_2 \\ &= \frac{1}{4K_{yy}}\left(K_{xx}K_{yy} - K^2_{xy}\right)dz_2 d\bar{z}_2 + \frac{1}{4}K_{yy}\left|dz_1 - i\frac{K_{xy}}{K_{yy}}dz_2\right|^2 . \end{aligned} \tag{4.12}$$

Performing the Legendre transformation and using (4.3) we obtain

$$ds^2 = \frac{1}{4}\tilde{K}_{xx}(dx^2 + d\theta^2) - \frac{1}{4\tilde{K}_{\eta\eta}}\left[\left(dy + \tilde{K}_{x\eta}dx\right)^2 + \left(d\phi - \tilde{K}_{x\eta}d\theta\right)^2\right]. \tag{4.13}$$

We can change coordinates $y \to \eta$ noting that $dy + \tilde{K}_{x\eta}dx = -\tilde{K}_{\eta\eta}d\eta$ to get

$$ds^2 = -\frac{1}{4}\tilde{K}_{\eta\eta}\left[x(dx^2 + d\theta^2) + d\eta^2\right] - \frac{1}{4\tilde{K}_{\eta\eta}}\left(d\phi - \tilde{K}_{\eta x}d\theta\right)^2 , \tag{4.14}$$

where we used eq. (4.7) for $\tilde{K}$. One simple check is that the solution (4.8) with $a_0 = 2/3$ gives the metric

$$ds^2 = x(dx^2 + d\theta^2) + d\eta^2 + d\phi^2 \,, \tag{4.15}$$

which is the same as the metric (2.1) after choosing $\Omega = 1/z$, $\tau = i \ln z$ and $z = \exp(x + i\theta)$.

Another thing we need to ensure is that the metric is positive definite. As it stands, it is clear that it would be positive definite as long as $\tilde{K}_{\eta\eta} < 0$. To check that notice that (4.9) can also be written as

$$\tilde{K}(\eta, x) = a_0(x^3 - 3\eta^2) - \frac{3^{1/3}}{2\pi^2}\Gamma\left(\frac{2}{3}\right)^3 \int_{-\infty}^{+\infty} \frac{x}{\left[\tilde{\eta}^2 + \frac{4}{9}x^3\right]^{5/6}} h(\tilde{\eta} + \eta) d\tilde{\eta} \,, \tag{4.16}$$

which shows that $\tilde{K}_{\eta\eta}$ is given by

$$\tilde{K}(\eta, x)_{\eta\eta} = -6a_0 - \frac{3^{1/3}}{2\pi^2}\Gamma\left(\frac{2}{3}\right)^3 \int_{-\infty}^{+\infty} \frac{x}{\left[\tilde{\eta}^2 + \frac{4}{9}x^3\right]^{5/6}} h_{\eta\eta}(\tilde{\eta} + \eta) d\tilde{\eta} \,, \tag{4.17}$$

which is not surprising since $\tilde{K}_{\eta\eta}$ satisfies the same equation (4.7). The point, however, is that we can rewrite this as

$$\tilde{K}(\eta, x)_{\eta\eta} = -\frac{3^{1/3}}{2\pi^2}\Gamma\left(\frac{2}{3}\right)^3 \int_{-\infty}^{+\infty} \frac{x}{\left[\tilde{\eta}^2 + \frac{4}{9}x^3\right]^{5/6}} (6a_0 + h_{\eta\eta}(\tilde{\eta} + \eta)) d\tilde{\eta} \,, \tag{4.18}$$

so, as long as $\tilde{K}_{\eta\eta}$ is negative at $x = 0$:

$$\tilde{K}_{\eta\eta}(x = 0, \eta) = -(6a_0 + h_{\eta\eta}(\eta)) < 0 \,, \tag{4.19}$$

then $\tilde{K}_{\eta\eta} < 0$ for any value of x since the integrand in (4.18) is positive. That means that as long as we choose $h(\eta)$ appropriately, namely satisfying (4.19), the metric will be positive definite.

As a further verification, we can start with the following ansatz

$$\begin{aligned} ds^2 &= x\,f\,(dx^2 + d\theta^2) + f d\eta^2 + \frac{1}{f}(d\phi + B d\theta)^2 \\ \phi &= -\ln(Nx) \,, \\ \chi &= N\theta \,, \\ \tau &= \chi + ie^{-\phi} = N(\theta + ix) \,, \end{aligned} \tag{4.20}$$

and verify that the equations of motion (3.1) are satisfied if

$$f_{xx} + x f_{\eta\eta} = 0 \,, \qquad B_\eta = -f_x \,, \qquad B_x = x f_\eta \,. \tag{4.21}$$

Identifying $f = -\tilde{K}_{\eta\eta}$ and $B = \tilde{K}_{\eta x}$ we see that these last equations are implied by eq. (4.7). In (4.20) we used the rescaling freedom we mention in eq. (3.4) to scale away the 1/4 in the metric and to introduce a parameter N that we associate with the number of $D7$-branes since now, when $\theta \to \theta + 2\pi$ we get $\chi \to \chi + 2\pi N$.

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It is useful to compute the Ricci scalar for the metric (4.20) which is

$$R = \frac{1}{f x^3}\,. \tag{4.22}$$

We see that, as long as $f > 0$ the only singularity is at $x = 0$. This singularity is the same as the one that appears for the flat brane. Another fact that we can check using the curvature is that this solution is not just a change of coordinates of the usual flat brane. The flat brane corresponds to constant f. We can write f as

$$f = \frac{1}{x^3 R} = N^3 e^{3\phi}\frac{1}{R}\,. \tag{4.23}$$

If f is constant, we have

$$\nabla_\mu \frac{e^{3\phi}}{R} = 0\,, \tag{4.24}$$

which is a coordinate independent statement. When f is not constant the derivative does not vanish and so the background is different. That is to say, it might be possible to redefine the coordinates so that the metrics agree but then the functions ϕ will differ.

As a final check we can use the construction in eq. (3.19) and find a Ricci flat metric

$$ds^2 = f\left(x(dx^2+d\theta^2)+d\eta^2\right)+\frac{1}{f}\left(d\phi+Bd\theta\right)^2+\frac{1}{x}\left(d\alpha-\frac{\beta}{x}d\theta\right)^2+\frac{1}{x}\left(d\beta-\frac{\beta}{x}dx\right)^2\,, \tag{4.25}$$

where we introduced real coordinates α and β through

$$\zeta = \alpha + i\beta\,. \tag{4.26}$$

A straightforward calculation shows that, given (4.21), the metric (4.25) is Ricci flat.

The surfaces of constant dilaton and axion are the surfaces of constant $x = x_0$ and θ. They are parameterized by η and ϕ with an induced metric given by

$$ds^2_{\text{ind.}} = f(x_0,\eta)d\eta^2 + \frac{1}{f(x_0,\eta)}d\phi^2\,, \tag{4.27}$$

Since this is the metric induced on the surfaces of constant dilaton and axion, it characterizes the geometry around the D7 brane in a coordinate independent way and therefore should be related to the metric of the manifold where the D7 brane is wrapped. The precise relation is worked out after eq. (5.13) in a more general case.[4] By construction, the metric (4.27) has an isometry $\phi \to \phi + \alpha$. The topology depends on the function $f(x_0,\eta)$. If $f > 0$ for all values of η, $-\infty < \eta < \infty$ and $0 < \lim_{\eta\to\pm\infty} f(x_0,\eta) < \infty$ then the manifold is non compact of topology $\mathbb{R}^1 \times S^1$. One can get a topology S^2 if the circle ϕ closes which means that f diverges at some point. For example the standard spherical round metric is obtained for $f = 1/(1-\eta^2)$ as can be seen by replacing $\eta = \cos\Theta$. We see however

[4] If we use the same procedure that leads to eq. (5.13) we get the same metric (4.27) up to a factor $(1+\frac{B^2}{xf^2})^{-1}$. This factor is non-singular for $x \neq 0$ and therefore the topological properties of the manifold where the $D7$ brane is wrapped are the same as those of the manifolds with constant axidilaton that we analyze here

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that, if we use such a function f, the metric (4.20) will be singular at $\eta = \pm 1$ for any x since f also multiplies $dx^2 + d\theta^2$. Therefore, the topology S^2 cannot be obtained from this ansatz. However we can still obtain a compact manifold if f is periodic in η. In that case we can periodically identify η with the same period and obtain a manifold whose topology is T^2 with a non-flat metric given by (4.27). The solution in this case can be obtained by considering a generic periodic function

$$h(\eta) = \sum_{n=1}^{\infty} a_n \cos(n\eta) \,, \tag{4.28}$$

where the a_n are arbitrary coefficients restricted only by the fact that h, h_η and $h_{\eta\eta}$ should be bounded continuous functions. Replacing in (4.9) gives

$$\tilde{K} = a_0(x^3 - 3\eta^2) - 3^{2/3}\Gamma\left(\frac{2}{3}\right) \sum_{n=1}^{\infty} a_n \mathbf{Ai}\Big(x n^{3/2}\Big) \cos(n\eta) \,, \tag{4.29}$$

where $\mathbf{Ai}$ denotes the Airy function that vanishes at infinity and a_0 should be chosen so as to ensure that $\tilde{K}_{\eta\eta}(x=0,\eta) < 0$ i.e. $6a_0 + h_{\eta\eta} > 0$. The result can also be written using Bessel functions $\mathbf{K}_\nu$ since

$$\mathbf{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}\mathbf{K}_{1/3}\left(\frac{2}{3}x^{3/2}\right) . \tag{4.30}$$

As an example we can take

$$\tilde{K} = x^3 - 3\eta^2 - 3^{2/3}\Gamma\left(\frac{2}{3}\right) \mathbf{Ai}(x)\cos(\eta) \,. \tag{4.31}$$

Since the Airy function vanishes exponentially at infinity, we see that for $x \to \infty$ we have $\tilde{K} \simeq a_0(x^3 - 3\eta^2)$, that is the metric of the flat D7 brane. All the influence of the coefficients $a_{n \geq 1}$ disappears. This would suggest that large x corresponds to the UV properties of the theory on the D7 branes since at short distances the details of the metric are irrelevant.

5. Complex dimension 2 (4 real supercharges)

In this section we study the case when $p = 2$, that is when we wrap the D7 brane on a manifold of dimension $d = 4$. In this case we were not able to linearize the equation as in the case $p = 1$ ($d = 2$). Again we have to solve

$$\det \partial_{a\bar{b}} K = \Omega(z_c)\bar{\Omega}(z_{\bar{c}})\tau_2 \,, \tag{5.1}$$

where now $a, \bar{b} = 1 \ldots 3$. We can always take $\tau = z_3$. If we now assume that K is only a function of τ_2 we can simplify the equation. If we define $z_3 = \theta + ix$ similarly as before, then eq. (5.1) can be written as

$$\det \begin{pmatrix} K_{a\bar{b}} & -\frac{1}{2i}K_{ax} \\ \frac{1}{2i}K_{\bar{a}x} & \frac{1}{4}K_{xx} \end{pmatrix} = \frac{1}{4}K_{xx} \det\left(K_{a\bar{b}} - \frac{K_{ax}K_{\bar{b}x}}{K_{xx}}\right) = \Omega(z_c)\bar{\Omega}(z_{\bar{c}})x \,, \tag{5.2}$$

where now $a, \bar{b} = 1, 2$ since we wrote the dependence in z_3 explicitly. Now we can do a Legendre transformation from x to η. Notice that this is different from what we did before where the Legendre transformation was not with respect to x. In any case the calculations are similar. We define

$$\tilde{K} = K - \eta x\,, \qquad \eta = K_x\,, \tag{5.3}$$

which results in

$$K_a = \tilde{K}_a\,, \quad \tilde{K}_\eta = -x\,, \quad K_{a\bar{b}} = \tilde{K}_{a\bar{b}} - \frac{\tilde{K}_{a\eta}\tilde{K}_{\bar{b}\eta}}{\tilde{K}_{\eta\eta}}\,, \quad K_{xx} = -\frac{1}{\tilde{K}_{\eta\eta}}\,, \quad K_{ax} = -\frac{\tilde{K}_{a\eta}}{\tilde{K}_{\eta\eta}}\,. \tag{5.4}$$

Replacing in (5.2) we get

$$\det(\tilde{K}_{a\bar{b}}) = 4|\Omega|^2 \tilde{K}_\eta \tilde{K}_{\eta\eta}\,, \tag{5.5}$$

where we are assuming that Ω is independent of τ. Since the equation did not linearize we can only search just for a particular solution rather than solving it in general. In order to do that we use a factorized ansatz:

$$\tilde{K} = \Phi(z^a, z^{\bar{a}})X(\eta) \qquad \Rightarrow \qquad X^2 \det(\Phi_{a\bar{b}}) = 4\Phi^2 X_\eta X_{\eta\eta}\,. \tag{5.6}$$

With this ansatz we get two equations:

$$X^2 = \pm 4 X_\eta X_{\eta\eta} = \pm\frac{4}{3}\frac{d}{dX}X_\eta^3 \tag{5.7}$$

$$\det \Phi_{a\bar{b}} = \pm\Phi^2\,. \tag{5.8}$$

The first equation can be easily integrated giving

$$\eta = \int^X \frac{d\tilde{X}}{(A \pm \frac{\tilde{X}^3}{4})^{\frac{1}{3}}}\,, \tag{5.9}$$

where A is an integration constant. Before looking into the equation for Φ we find out what is the metric like under the assumptions we have made so far. We have

$$ds^2 = K_{a\bar{b}}dz^a d\bar{z}^{\bar{b}} - \frac{1}{2i}K_{ax}dz^a d\bar{z}_3 + K_{\bar{a}x}\frac{1}{2i}d\bar{z}^{\bar{a}}dz_3 + \frac{1}{4}K_{xx}dz_3 d\bar{z}_3\,. \tag{5.10}$$

With the definition $z_3 = \theta + ix$, and after some algebra we arrive at

$$ds^2 = \left(K_{a\bar{b}} - \frac{K_{ax}K_{\bar{b}x}}{K_{xx}}\right)dz^a d\bar{z}^{\bar{b}} + \frac{1}{4K_{xx}}(dK_x)^2 + \frac{K_{xx}}{4}\left(d\theta + \frac{i}{K_{xx}}(K_{ax}dz^a - K_{\bar{a}x}d\bar{z}^{\bar{a}})\right)^2. \tag{5.11}$$

Now we transform coordinates from x to η and introduce the Legendre transform to get

$$ds^2 = \tilde{K}_{a\bar{b}}dz^a d\bar{z}^{\bar{b}} - \frac{1}{4}\tilde{K}_{\eta\eta}d\eta^2 - \frac{1}{4\tilde{K}_{\eta\eta}}\left(d\theta + i\left(\tilde{K}_{a\eta}dz^a - \tilde{K}_{\bar{a}\eta}d\bar{z}^{\bar{a}}\right)\right)^2. \tag{5.12}$$

The ansatz $\tilde{K} = \Phi X$ gives now

$$ds^2 = X\Phi_{a\bar{b}}dz^a d\bar{z}^{\bar{b}} - \frac{1}{4}\Phi X_{\eta\eta}d\eta^2 - \frac{1}{4X_{\eta\eta}\Phi}\left(d\theta - iX_\eta\left(\Phi_a dz^a - \Phi_{\bar{a}}d\bar{z}^{\bar{a}}\right)\right)^2. \tag{5.13}$$

We can consider η as a radial variable. For fix η, there is a four dimensional Kähler manifold with Kähler potential $\Phi(z^a, z^{\bar{b}})$ whose volume is proportional to $X(\eta)^4$ and over which a circle parameterized by θ is fibered. The metric of the manifold where the $D7$

brane is wrapped is then a Kähler metric with Kähler potential Φ and the S^1 fibration is the supergravity equivalent of the $\mathrm{U}(1)_R$ twist that we discussed at the beginning should be introduced in the theory on the brane to preserve supersymmetry. Another useful metric to consider is that of the surfaces of constant dilaton and axion that we discussed in the previous section. Here that means surfaces of constant z_3. Their metric is given by

$$ds^2_{\text{ind}} = K_{a\bar{b}} dz^a d\bar{z}^{\bar{b}} . \tag{5.14}$$

Basically, we can say that the surfaces of constant axidilaton have a Kähler potential K (for fixed z_3) and the manifold where the $D7$-brane is wrapped a Kähler potential $\tilde{K}$ (for fixed η), where $\tilde{K}$ is the Legendre transform of K with respect to $x = \tau_2$.

Now we have to find a solution to the equation for Φ. To do this, we assume that Φ is a function of $\rho = z_1\bar{z}_1 + z_2\bar{z}_2$ which introduces an SU(2) isometry in the metric. We get the equation

$$\det(\Phi_{a\bar{b}}) = \pm\Phi^2 \qquad \Rightarrow \qquad \Phi_\rho^2 + \rho\Phi_\rho\Phi_{\rho\rho} = \pm\Phi^2 . \tag{5.15}$$

This equation is homogeneous in Φ which suggests a change of variable $\Phi = \exp(f)$ resulting in

$$f'^2 + \rho f'(f'' + f'^2) = \pm 1 , \tag{5.16}$$

we now define $y = f'$ and get a first order equation

$$\rho y y' = \pm 1 - \rho y^3 - y^2 . \tag{5.17}$$

This type of equation is known as an Abel equation but unfortunately, to our knowledge, it does not have a solution in terms of elementary functions. It is easy however to find a solution as a series expansion or numerically. Before doing that we rewrite the metric (5.13) to understand what choice of sign to do and which boundary conditions to impose on $y(\rho)$.

Using that Φ is only a function of ρ, the metric (5.13) can be written as

$$\begin{aligned} ds^2 &= X\left(\Phi_\rho dz^a d\bar{z}^{\bar{a}} + \Phi_{\rho\rho}\bar{z}^a dz^a\, z^{\bar{b}} d\bar{z}^{\bar{b}}\right) - \frac{1}{4}\Phi X_{\eta\eta} d\eta^2 - \\ &\quad - \frac{1}{4X_{\eta\eta}\Phi}\left(d\theta - iX_\eta\Phi_\rho\left(\bar{z}^a dz^a - z^{\bar{a}} d\bar{z}^{\bar{a}}\right)\right)^2 . \end{aligned} \tag{5.18}$$

It is now convenient to change into angular variables ϑ, ψ, ϕ:

$$z_1 = \sqrt{\rho}\cos\frac{\vartheta}{2} e^{\frac{1}{2}i(\psi+\phi)} , \qquad z_2 = \sqrt{\rho}\sin\frac{\vartheta}{2} e^{\frac{1}{2}i(\psi-\phi)} , \tag{5.19}$$

and introduce the 1-forms σ_1, σ_2, σ_3 through:

$$\sigma_1 + i\sigma_2 = e^{-i\psi}(d\vartheta + i\sin\vartheta d\phi) , \qquad \sigma_3 = d\psi + \cos\vartheta\, d\phi , \tag{5.20}$$

which result in

$$\begin{aligned} |dz_1|^2 + |dz_2|^2 &= \frac{1}{4\rho} d\rho^2 + \frac{\rho}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) , \\ |\bar{z}_1 dz_1 + \bar{z}_2 dz_2|^2 &= \frac{1}{4} d\rho^2 + \frac{1}{4}\rho^2\sigma_3^2 , \\ z^{\bar{a}} dz^a - z^a d\bar{z}^{\bar{a}} &= i\rho\sigma_3 . \end{aligned} \tag{5.21}$$

Finally, it is convenient to change coordinates from η to X using that from equation (5.9) we have:

$$X_\eta = \left(A \pm \frac{X^3}{4}\right)^{1/3}, \qquad X_{\eta\eta} = \frac{X^2}{4}\left(A \pm \frac{X^3}{4}\right)^{-1/3}. \tag{5.22}$$

With these results and eqns. (5.21), the metric (5.18) can be written as

$$\begin{aligned} ds^2 =& \pm\frac{X}{y}e^f\frac{d\rho^2}{4\rho} + \frac{\rho X}{4}ye^f(\sigma_1^2+\sigma_2^2) \pm \frac{\rho X}{4y}e^f\sigma_3^2 \mp \frac{1}{16}e^f\frac{X^2}{(A\pm X^3/4)}dX^2 \mp \\ &\mp \frac{(A\pm X^3/4)^{1/3}}{X^2}\left(d\theta + \left(A\pm\frac{X^3}{4}\right)^{1/3}\rho y e^f\sigma_3\right)^2. \end{aligned} \tag{5.23}$$

If we choose X to be positive then, to get a positive definite metric we have to choose the upper signs everywhere. However, this has to be complemented by taking $A < 0$ and $0 < \frac{X^3}{4} < A$. Up to a rescaling, we can take $A = 1/4$, $0 < X < 1$. It is also convenient to introduce a new radial coordinate $u = \sqrt{\rho}$. The final form of the solution is

$$\begin{aligned} ds^2 =& Xe^f\left[\frac{1}{y}du^2 + \frac{1}{4}u^2\left(y(\sigma_1^2+\sigma_2^2) + \frac{1}{y}\sigma_3^2\right)\right] + \frac{1}{4}e^f\frac{X^2}{1-X^3}dX^2 + \\ &+4^{-1/3}e^{-f}\frac{(1-X^3)^{1/3}}{X^2}\left(d\theta - 4^{-1/3}u^2ye^f(1-X^3)^{1/3}\sigma_3\right)^2, \\ \phi =& -\ln\left(N4^{-1/3}e^f(1-X^3)^{1/3}\right), \\ \chi =& N\theta, \end{aligned} \tag{5.24}$$

where we remind the reader that $y(\rho)$ is a function satisfying (5.17) with a plus sign. Also, $y = \frac{df}{d\rho}$ and $\rho = u^2$. The values of ϕ and χ follow from the initial ansatz:

$$\tau = \chi + ie^{-\phi} = z_3 = \theta + ix = \theta - i\tilde{K}_\eta = \theta - i\Phi X_\eta = \theta + i4^{-\frac{1}{3}}e^f(1-X^3)^{\frac{1}{3}}. \tag{5.25}$$

In the metric (5.24), the manifold spanned by $(u, \vartheta, \psi, \phi)$ closes smoothly at $u = 0$ only if $y(u = 0) = 1$. This is because $d\Omega_{[3]}^2 = (\sigma_1^2+\sigma_2^2+\sigma_3^2)/4$ is the metric of a round three sphere. In fact, the equation (5.17) that y satisfies actually implies that the only solution regular at $\rho = 0$ satisfies precisely $y(0) = 1$. We can get a series expansion as

$$y(\rho) = 1 - \frac{1}{3}\rho + \frac{7}{36}\rho^2 - \frac{16}{135}\rho^3 + \frac{931}{12960}\rho^4 - \frac{163}{3780}\rho^5 + \cdots. \tag{5.26}$$

The asymptotic behavior follows from the same equation (5.17) and is given by

$$y(\rho) \simeq \rho^{-1/3}, \qquad \text{for} \qquad \rho \to \infty. \tag{5.27}$$

With this solution for $y(\rho)$, the manifold over which the $D7$-brane is wrapped, namely the one spanned by $(u, \vartheta, \psi, \phi)$ is non-singular and also non-compact. On the other hand, the full metric is singular at $X = 0$ and $X = 1$. This is worse than the previous case where the metric was singular only at the 'core' where $\tau_2 = 0$ which here would be $X = 1$. It is not clear to us the interpretation of the extra singularity at $X = 0$.

It is instructive to perform a similar calculation as in this section for the case $p = 1$. This also gives two singularities in the radial direction as opposed to what we obtained in the previous section, namely only a singularity at $x = 0$. This indicates that perhaps an improvement of the method used in this section can lead to avoid the singularity at $X = 0$ but we did no explore that. For completeness, we write down the solution for $p = 1$ that we just mentioned:

$$
\begin{aligned}
ds^2 &= \sin u \cosh^{4/3}\chi \, d\vartheta^2 + \frac{4}{9} \sin u \cosh^{2/3}\chi \, \left(d\chi^2 + du^2\right) + \\
&\quad + \frac{\cos^{2/3}u}{\cosh^{2/3}\chi \sin u} \left(d\theta + \sinh\chi \cos^{2/3}u \, d\vartheta\right)^2 , \\
\tau &= \theta + i \left(\cosh\chi \cos u\right)^{2/3} ,
\end{aligned}
\tag{5.28}
$$

where the variables run over: $0 < u < \pi/2$, $0 < \chi < \infty$, $0 < \vartheta < 2\pi$, $0 < \theta < 2\pi$. The metric is singular at $u = 0$ and $u = \pi/2$.

Finally let us mention that from the solution (5.24) we can get, using (3.19), a 6-dimensional Ricci flat metric.

6. Generalizations

The solutions we have obtained up to now are generalizations of the "circularly symmetric" D7-brane of [7]. These solutions have a curvature singularity where $\tau_2 = 0$. This follows from the equation for the metric (3.3):

$$
R_{\mu\nu} = \frac{1}{4\tau_2^2} \left(\partial_\mu \tau \partial_\nu \bar{\tau} + \partial_\nu \tau \partial_\mu \bar{\tau}\right) . \tag{6.1}
$$

Instead, in the original $D7$-brane construction [5, 6] a solution was proposed where $\tau_2 > 0$ everywhere. That was accomplished by using the freedom in choosing the functions Ω, τ in the solution (2.1). One chooses a holomorphic function $\tau(z) = \tau_1 + i\tau_2$ that maps the complex plane onto the fundamental domain: $-1/2 \leq \tau_1 \leq 1/2$, $\tau_2 > 0$, $|\tau| \geq 1$. Such a function has cuts where τ jumps by an $\mathrm{SL}(2, \mathbb{Z})$ transformation. These transformations now include, besides $\tau \to \tau + 1$ also $\tau \to -1/\tau$ and combinations. In that case $\tau_2(z, \bar{z})$ also has cuts and cannot enter directly in the metric. However, choosing Ω adequately one can make the combination $\Omega(z)\, \bar{\Omega}(\bar{z})\, \tau_2(z, \bar{z})$ that appears in the metric invariant under modular transformations and therefore, a function of z, $\bar{z}$ with no cuts. In spite of the fact that $\tau_2 > 0$ everywhere, the resulting solution still has curvature singularities because $\partial_\mu \tau$ diverges at points where $\tau = i$ or $\tau = e^{i\pi/3}$. This singularities are milder since at a distance δ from the singularity $\partial_\mu \tau \sim \delta^{-\alpha}$, $\alpha < 1$ and therefore the action obtained by integrating the Ricci scalar on a transverse plane is finite.

In this section we generalize this type of solution to a $D7$ brane wrapped on a 2 dimensional manifold using the same calculations as in section 4. We are looking then for a function $K(z_1, z_2, \bar{z}_1, \bar{z}_2)$ such that

$$
K_{1\bar{1}} K_{2\bar{2}} - K_{1\bar{2}} K_{2\bar{1}} = |\Omega|^2 \tau_2 . \tag{6.2}
$$

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Now assume that K is only a function of the real part of z_1 and introduce real coordinates through

$$z_1 = y + i\phi \,. \tag{6.3}$$

Furthermore we then assume that τ and Ω are functions only of z_2. Thus, eq. (6.2) becomes

$$K_{yy}K_{2\bar{2}} - K_{y2}K_{y\bar{2}} = 4|\Omega(z_2)|^2 \frac{\tau(z_2) - \bar{\tau}(\bar{z}_2)}{2i} \,. \tag{6.4}$$

The function $\tau(z_2)$ can have cuts but Ω is then chosen such that the right hand side of the equation is well defined throughout the z_2 plane. Now, we use again a Legendre transformation

$$\tilde{K}(\eta, z_2, \bar{z}_2) = K(y, z_2, \bar{z}_2) - \eta y \,, \qquad \eta = K_y \,, \tag{6.5}$$

which results in

$$y = -\tilde{K}_\eta \,, \quad K_{2y} = -\frac{\tilde{K}_{2\eta}}{\tilde{K}_{\eta\eta}} \,, \quad K_{2\bar{2}} = \tilde{K}_{2\bar{2}} - \frac{\tilde{K}_{2\eta}\tilde{K}_{\bar{2}\eta}}{\tilde{K}_{\eta\eta}} \,, \quad K_{yy} = -\frac{1}{\tilde{K}_{\eta\eta}} \,. \tag{6.6}$$

Using this we linearize eq. (6.2):

$$\tilde{K}_{2\bar{2}} + 4\,\Omega\,\bar{\Omega}\,\tau_2\,\tilde{K}_{\eta\eta} = 0 \,. \tag{6.7}$$

The assumption that Ω is independent of y is important since $y = -\tilde{K}_\eta$ and any y dependence will render the equation non-linear. The metric is

$$\begin{aligned} ds^2 &= \tilde{K}_{2\bar{2}} dz_2 d\bar{z}_2 - \frac{1}{4\tilde{K}_{\eta\eta}} \left| dz_1 + 2\tilde{K}_{2\eta} dz_2 \right|^2 \\ &= \tilde{K}_{2\bar{2}} dz_2 d\bar{z}_2 - \frac{\tilde{K}_{\eta\eta}}{4} d\eta^2 - \frac{1}{4\tilde{K}_{\eta\eta}} \left(d\phi - i \left(\tilde{K}_{2\eta} dz_2 - \tilde{K}_{\bar{2}\eta} d\bar{z}_2 \right) \right)^2 \,. \end{aligned} \tag{6.8}$$

We still have to solve (6.7). Using separation of variables we can write a generic solution as

$$\tilde{K} = \int_{-\infty}^{+\infty} dk\, e^{ik\eta}\, \Phi_k(z_2, \bar{z}_2) \,, \tag{6.9}$$

where

$$\partial_2 \bar{\partial}_{\bar{2}} \Phi_k = 4\,k^2\,\Omega\,\bar{\Omega}\,\tau_2\,\Phi_k \,. \tag{6.10}$$

If we use the same Ω as in [5], this becomes

$$\partial_2 \bar{\partial}_{\bar{2}} \Phi_k = 4k^2\,\eta^2(z_2)\,\bar{\eta}^2(\bar{z}_2)\,\tau_2\,\Phi_k \,, \tag{6.11}$$

where $\eta(z)$ is Dedekinds η function:

$$\eta = q^{1/24} \prod_n (1 - q^n) \,, \tag{6.12}$$

with $q = e^{2\pi i \tau(z_2)}$ and $\tau(z_2)$ is defined implicitly through the equation

$$j(\tau) = z_2 \,, \tag{6.13}$$

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where j can be written in terms of Jacobi θ-functions as:

$$j(\tau) = \frac{\left(\theta_2^8(\tau) + \theta_3^8(\tau) + \theta_4^8(\tau)\right)^3}{\eta^{24}(\tau)}\,, \tag{6.14}$$

and has the virtue of being modular invariant. With these values of Ω and τ we should solve eq. (6.10):

$$\partial_{2\bar{2}}\Phi_k = 4\,k^2\,\eta^2\,\bar{\eta}^2\,\tau_2\,\Phi_k\,. \tag{6.15}$$

Unfortunately we were not able to find a solution to this equation so we leave the metric as it is in terms of the function Φ_k that should be determined, perhaps numerically.

Another possible generalization which we did not consider is to introduce D3 branes parallel to the D7 branes along the directions that we did not wrap. It seems possible that one can find such solutions along the lines of [12].

7. Conclusions

We described a generalization of the D7 brane solution such that the world volume of the brane is $\mathbb{R}^{[1,5]} \times T^2$ where the metric on T^2 is an arbitrary metric with translation invariance along one of the directions of the torus. The solution has a singularity at a finite radius in the same way as the flat brane has. Also, the function τ has a cut with a jump of the form $\tau \to \tau + N$ with N an integer counting the number of branes. In analogy with the flat brane case, we further discussed how we can make this singularity milder by introducing additional cuts in the function τ where τ is identified up to an $\mathrm{SL}(2,\mathbb{Z})$ transformation including $\tau \to -1/\tau$. In this case we did not write the metric explicitly but left it expressed in terms of the solutions of a certain linear equation.

All these supergravity backgrounds were obtained by using the same Legendre transform method which is sometimes used to find 4-dimensional hyperKähler metrics.

We also considered the case where we wrapped the brane on a 4-dimensional manifold. In this case the equation cannot be simplified as much as in the previous case and we find only a solution that is singular close and far from the brane. The interpretation of this solution is not clear.

We would like to point out that the solution with the D7-brane wrapped on T^2 depends on an arbitrary function that determines the metric on T^2. Recently, it has been suggested that it might be possible to obtain solutions for black p-branes where the horizon is not uniform [13]. In our case there is no horizon but a singularity. However, it would be interesting to see if by applying T and S-dualities to our solution one can shed some light on the problem of non uniform branes.

Another point that would be interesting to pursue is the inclusion of D3 branes in this set up. This would lead to field theories with fields in the fundamental of the type discussed in detail in [14]. By immersing the D7 branes in AdS_5 the gauge group on the D7 branes becomes a global (flavor) group in the boundary theory.

Finally, it would also be interesting to understand the relation, if any, to work done in the case of $M5$-branes [15, 16]. Those solutions were mostly for branes wrapped on manifolds of constant curvature but could be generalized to a situation similar to the one analyzed here.

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A. General solution to the equation for $\tilde{K}(x,\eta)$

In this appendix we solve the equation

$$\tilde{K}_{xx} + x\tilde{K}_{\eta\eta} = 0\,. \tag{A.1}$$

The equation has similar properties to the Laplace equation. The variable's domain of variation is: $-\infty < \eta < \infty$, $0 \le x < \infty$. We are going to use as boundary conditions

$$\tilde{K}(x,\eta)\Big|_{x=0} = h(\eta)\,, \tag{A.2}$$

and that $\tilde{K}$ remains finite for $x \to \infty$. Since the equation is linear, it can be trivially solved by separation of variables with the result

$$\tilde{K} = -3^{2/3}\Gamma\left(\frac{2}{3}\right)\int_{-\infty}^{+\infty} dk\, h(k) e^{ik\eta}\mathbf{Ai}\left(k^{2/3}x\right)\,, \tag{A.3}$$

where $\mathbf{Ai}$ is the Airy function[5] which solves the equation

$$\frac{d^2\mathbf{Ai}(x)}{dx^2} - x\mathbf{Ai}(x) = 0\,, \tag{A.4}$$

and vanishes at infinity. It can be written in terms of the Bessel function $\mathbf{K}_{\frac{1}{3}}$ as:

$$\mathbf{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}\mathbf{K}_{1/3}\left(\frac{2}{3}x^{3/2}\right)\,. \tag{A.5}$$

There is an independent solution that blows up as $x \to \infty$ and therefore should be discarded since we want $\tilde{K}$ to be finite. In $\tilde{K}$ we included a (negative) constant coefficient in front because the function $\mathbf{Ai}$ is normalized such that

$$\mathbf{Ai}(0) = \frac{1}{3^{2/3}\Gamma(2/3)}\,. \tag{A.6}$$

[5]In this appendix we use several properties of special functions which can be obtained from [17] or by using a computer algebra program such as Maple v.9

This means that

$$\tilde{K}(x=0,\eta) = -h(\eta) = -\int_{-\infty}^{+\infty} h(k)e^{ik\eta}\,, \tag{A.7}$$

namely $h(k)$ is the Fourier transform of $h(\eta)$. Inverting this we arrive at

$$\tilde{K} = -3^{2/3}\Gamma\left(\frac{2}{3}\right)\int_{-\infty}^{+\infty} dk\,\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\tilde{\eta}\,e^{-ik\tilde{\eta}}h(\tilde{\eta})e^{ik\eta}\mathbf{Ai}\left(k^{2/3}x\right). \tag{A.8}$$

Interchanging the order of integration and performing the k integral we obtain

$$\tilde{K}(\eta,x) = a_0(x^3-3\eta^2) - \frac{3^{1/3}}{2\pi^2}\Gamma\left(\frac{2}{3}\right)^3\int_{-\infty}^{+\infty}\frac{x}{\left[(\tilde{\eta}-\eta)^2+\frac{4}{9}x^3\right]^{5/6}}h(\tilde{\eta})d\tilde{\eta}\,, \tag{A.9}$$

where, for completeness, we added a trivial solution $a_0(x^3-3\eta^2)$ to what we obtained from (A.8). The last expression is used in the main text.

B. Some properties of Kähler manifolds

In this appendix we summarize some well known properties of Kähler manifolds which are useful in the main text. On a Kähler manifold, the metric can be written through a Kähler potential as

$$ds^2 = \partial_{a\bar{b}}K dz^a d\bar{z}^{\bar{b}}\,. \tag{B.1}$$

The non-vanishing components of the Levi-Civita connection can be computed using the usual definition and are given by

$$\Gamma^c_{ab} = g^{c\bar{d}}\partial_{ab\bar{d}}K\,, \qquad \Gamma^{\bar{c}}_{\bar{a}\bar{b}} = g^{\bar{c}d}\partial_{\bar{a}\bar{b}d}K\,. \tag{B.2}$$

From here, we obtain that the only non-vanishing components of the Ricci tensor are

$$R_{a\bar{b}} = -\partial_{\bar{b}}\Gamma^c_{ac} = -\partial_{\bar{b}}(g^{c\bar{d}}\partial_a g_{c\bar{d}}) = -\partial_{a\bar{b}}\ln\det g_{c\bar{d}}\,. \tag{B.3}$$

The determinant in the last expression is that of the matrix $g_{a\bar{b}}$ as indicated. Note that the determinant of the metric is: $\det g = (\det g_{a\bar{b}})^2$. One can introduce a vielbein: e^A_a, $\bar{e}^{\bar{A}}_{\bar{a}}$ such that

$$g_{a\bar{b}} = \partial_{a\bar{b}}K = e^C_a\bar{e}^{\bar{C}}_{\bar{b}}\,. \tag{B.4}$$

Deriving the last equality with respect to z^d and antisymmetrizing in a,d we obtain the relation

$$\partial_d e^C_a\bar{e}^{\bar{C}}_{\bar{b}} - \partial_a e^C_d\bar{e}^{\bar{C}}_{\bar{b}} = e^C_d\partial_a\bar{e}^{\bar{C}}_{\bar{b}} - e^C_a\partial_d\bar{e}^{\bar{C}}_{\bar{b}}\,. \tag{B.5}$$

Using this result in the definition of spin connection

$$\omega^{MNP} = \frac{1}{2}e^{nN}e^{mM}e^P_{[n,m]} - \frac{1}{2}e^{nP}e^{mM}e^N_{[n,m]} - \frac{1}{2}e^{mN}e^{nP}e^M_{[n,m]}\,, \tag{B.6}$$

we get that the only non-vanishing components of ω are

$$\omega^{CA\bar{B}} = -\omega^{C\bar{B}A} = -e^{a\bar{B}}e^{\bar{b}C}\partial_{\bar{b}}e^A_a\,, \tag{B.7}$$

and their complex conjugates. From here, we find that the U(1) part of the connection is given by

$$\sum_A \omega_{\bar{a}}{}^A{}_A = \sum_A \omega_{\bar{a}\bar{A}A} = -e^b_A \partial_{\bar{a}} e^A_b = -\partial_{\bar{a}} \ln\det e^A_a\,. \tag{B.8}$$

Notice also that

$$\det g_{a\bar{b}} = \det e^A_a \det \bar{e}^{\bar{B}}_{\bar{b}} = |\det e^A_a|^2\,. \tag{B.9}$$

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